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GENERAL THEORY OF DYNAMIC SYSTEMS AND CLASSICAL MECHANICS

A.N. Kolmogorov

Introduction.

I did not expect to deliver a lecture at this final session of our congress in this great hall with which I was more familiar as a place where musical masterworks conducted by Mengelberg had been performed. The lecture which I have prepared without regard for such a distinguished position on the agenda of our congress, will be devoted to a quite specialized set of problems. My task will be to elucidate the ways to use basic ideas and results of contemporary general metric and spectral theory of dynamic systems in the study of conservative dynamic systems of classical mechanics. However, it seems to me that the subject I have selected is of more general interest as an example of new unexpected and profound relations between various parts of classical and contemporary mathematics.

In his famous lecture delivered at the Congress of 1900, Gilbert said that the unity of mathematics, the impossibility of breaking it up into independent branches, arises from the very essence of our science. The most convincing corroboration of the correctness of this idea is the appearance of new junction points at each new stage of mathematical development where ideas and methods from the most diverse mathematical disciplines prove to be indispensable and enter into new linkage to solve totally new problems. One such junction point for nineteenth century mathematics has been in the field of problems of integration of differential equations systems of classical mechanics, where problems of mechanics and of differential equations have been organically interwoven with problems of variational calculus, of multi-dimensional differential geometry, the theory of analytic functions and the theory of continuous groups.

The fundamental role of topology problems in this cycle has become apparent after the work of Poincaré. On the other hand, the Poincaré-Karateodori theorem about the return, served as the beginning of the dynamic systems "metric" theory in the sense of studies of motion characteristics that take place during "almost all" initial aspects of this system. The subsequently evolving "ergodic theory" has been variously generalized and has become an independent center of gravity and linking point for methods and problems of various, especially recent branches of mathematics (the abstract measure theory, the theory of groups of linear operators in the Gilbert and other spaces of infinite dimensions, the stochastic process theory, etc.). A large report read by Kakutani (23) was devoted to the general problems of ergodic theory during the previous International Congress of 1950.

As is generally known, topological methods were significantly utilized in the oscillation theory and in particular, while solving fully concrete problems arising during the studies pertaining to automatic control systems in electrical engineering, etc. However, these tangible physical and technical uses pertain mainly to the nonconservative systems. The matter usually pertains to looking for separate asymptomatic, stable motions (in particular, stable, stationary points and stable boundary cycles), and to the studies of integral curves' bundles that are drawn to these asymptomatically stable motions.

The asymptomatically stable motions are impossible in conservative systems. Therefore, for example, looking for separate periodic motions, despite all its mathematical interest, is of only very limited, real, physical interest. As far as conservative systems are concerned, the metric point of view is of basic significance, as it makes possible to study the aspects of the basic mass of motions. The contemporary general ergodic theory has prepared for this purpose a collection of ideas possessing very great physical persuasiveness according to its intention. However, our progress up to the present time is more than limited, as far as the analysis of concrete problems of classic mechanics is concerned from these contemporary points of view.

In the first place, the matter concerns the following problem. Let us assume that motion according to the s-dimensional analytical variety V^s is being determined by the canonical system of differential equations with the Hamilton analytic function H $(q_1 \dots, q_s, p_1 \dots, p_s)$. Let us consider that there are k univalent analytical first integrals $I_1, I_2 \dots, I_k$ and assumptions pick out from the phrase space Ω^{2s}

$$I_1 = C_1, \dots, I_k = C_k$$

analytical variety M^{2s-k}. As is well known, with almost all values $C_1, \dots \dots C_k$ at M^{2s-k} there emerges an analytical invariant sensity in a natural way, which fact makes it possible to apply to motions at M^{2s-k} general principles of the metric theory of dynamic systems. It is natural to revert to these more modern means in cases where besides $I_1, \dots$ .. I_k, there are no single-valued analytical first integrals, or if finding them seems too difficult, and other classical analytical methods for terminating integration of the system prove to be also inapplicable. In such cases it is required, with the aid of these or other qualitative considerations, to solve the problem about this, whether the motion at M^{2s-k} will be transitive (i.e. will almost all M^{2s-k} consist of one unique ergodic set), to determine the character of the spectrum in the case of transitivity and in the case of the absence of transitivity, to learn with exactness to within the set of measure zero (or, if only correct to within the set of small measure) the character of the spectrum in these ergodic sets.

I know only two concrete problems of classical mechanics in which this program has been already solved to a greater or lesser degree:

1) For the motion under its own momentum along the closed surface V^2 of consistently negative curvature*) In 1939 Hopf determined that the motion over the three-dimensional sets L_h^3 isolated by the requirement of the power constancy $H = h$ is transitive and the spectrum is uninterrupted (see [8]).

2) As will be shown later, during the motion under its own momentum along the analytical surfaces close enough to the triaxal ellipsoid, the motion at L_h^3 is not transitive

*) Perhaps it would not be useless to note that in the ordinary equivalent space it is possible to assign a closed surface V^2 genus one and to distribute in its vicinity a final number of attraction or repulsion centers that develop at V^2 a potential of forces in such a manner that the mass point along the V^2 under the influence of introduced exterior forces will become mathematically equivalent to the inertial motion in the metric which posesses everywhere a negative curvature..

and, correct within the set of small measure, is divided into two-dimensional tori T^2, on each of which the motion is transitive and the spectrum is discrete (see end of Par. 2).

However, it seems to metthat now is exactly the time when a considerably more rapid forward motion would be possible.

Par. 1. Analytical Dynamic Systems and theis Stable Capacities.

Dynamic systems of classical mechanics serve as an individual case of analytical dynamic systems with an intbegral invariant. An analytical n-dimensional variety Ω^n (phase space of the system) is the bearer of such a dynamic system. According to this, admissible transformations of coordinates $x_1 \ldots . x_n$ of the point $x \epsilon \Omega^n$ will always be analytical.

The right parts of differential equations that determine the motion,

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, \ldots, x_n) \qquad (1)$$

and the invariant density that generated invariant dimension

$$m(A) = \int_A M(x) dx_1 \ldots dx_n$$

will be considered as analytical functions of coordinates.*)

*) Everywhere, when we simply speak about the dimension without further specialization, we imply m dimension.

In accordance with what has been said in the introduction we shall mainly deal with canonical systems, i.e. the systems with $n = 2s$, with division of coordinates of the point $(q, p) \in \Omega^{2s}$ into two groups $q_1, q_2, \ldots, q_s$ and $p_1, \ldots, p_s$ with transformations of tangency in the capacity of admissible transformation of coordinates with equations of canonical form..

$$\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q_\alpha} \tag{2}$$

and with invariant density

$$M(q, p) = 1.$$

Special attention will be paid to the problem as to what aspects of dynamic systems happen to be with the "arbitrary" F and M (or "arbitrary" function H(q, p) in the case of canonical systems) "typical" and which can appear only as an "exception". This problem, however, is very precise. The approach to this problem from the point of view of the category of corresponding sets in the functional spaces of the systems of of functions $\{F_\alpha, M\}$ (or functions H), regardless of well-known successes obtained in this direction in the general theory of absolute dynamic systems, is more interesting as a medium of the evidence of the existence, than as a direct answer to greatly conventionalized and idealized real enquiries made by physicists or mechanics. As far as the dimension is concerned, the approach to this problem, on the other hand, appears to be quite sound and natural from the physical point of view (as it has been argued in detail, for example, by Neumann (1)), but it runs against the absence of natural measure in functional spaces.

We shall follow two courses. In the first place, in order to obtain positive results that this or other type of dynamic system should be recognized as one of the significant, not "exclusive", ones and not a subject of "neglect" from any sound point of view (just as sets of measure zero are being neglected), we shall use the idea of stability in the sense of preserving a given type of behavior of dynamic system with a small modification of functions F_α and M, or functions H. Any type of behavior of a dynamic system existing with even one example of its stable realization, should be considered as an essential one and not ignored. In conformity with the accepted approach on the part of analytical functions, the "smallness" of the variation of function $f_o(x)$ will be understood in the sense of the transition from the function $f_o(x)$ to the function

$$f(x) = f_0(x) + \theta\varphi(x, \theta)$$

with the small value of parameter Θ and with the analyticity of function φ according to the totality of variables $x_1, x_2, \ldots, x_n, \theta$.

Such an approach to this matter mav incur criticism, though certain interesting results can be obtained with it. It will be shown that where one can confine oneself to the closeness of functions f_o and f in the sense of the closeness of their derivatives of restricted order.

In order to obtain negative results with unessential exclusive character of some phenomenon, we shall use only one, rather home-made kind of method: if in the class K of functions f(x) it is possible to introduce the final number of functionals

$$F_1(f),\ F_2(f), \ldots,\ F_r(f)$$

which in this or other sense are natural to consider as assuming "generally speaking arbitrary" values

$$F_1(f) = C_1, \ldots,\ F_r(f) = C_r$$

from some area of the r-dimensional space of points $C = (C_1, \ldots C_r)$, and we shall consider any phenomenon which can occur only in the presence of C out of the set which has an r-dimensional Lebesque measure zero as exclusive and subject to "neglect".

I shall begin a summary of concrete results obtained by applying this idea to the investigation of dynamic systems, whose phase space is a two-dimensional torus.

Par. 2. About Dynamic Systems on the Two-Dimensional Torus and Some Canonical Systems With Two Degrees of Freedom.

Like everywhere below, we shall consider the points of torus T^2 as defined circular coordinates x_1, x_2 (the point x does not vary during the transition from x_α to $x_\alpha + 2\pi$). Functions F_α located in the right side of the equations

$$\frac{dx_1}{dt} = F_1(x_1, x_2), \frac{dx_2}{dt} = F_2(x_1, x_2),$$

and the invariant density $M(x_1, x_2)$ we shall consider as analytical according to what was previously said, and furthermore we shall subject to conditions

$$F_1^2 + F_2^2 > 0, \ M > 0 \tag{1}$$

and to the condition of normalization $m(T^2) = 1$ for the sake of simplicity. We shall introduce mean frequencies of inversion

$$\lambda_1 = \int_{T^2} F_1(x)dm \ , \ \lambda_2 = \int_{T^2} F_2(x)dm.$$

A slight intensification of results obtained by Poincaré, Dazhua and Knezer leads, in this instance, to the conclusion that by the analytical transformation of the coordinates of the equation of motion it is possible to present them in the form

$$\frac{dx_1}{dt} = \lambda_1 M(x_1, x_2), \frac{dx_2}{dt} = \lambda_2 M(x_1, x_2).$$

It is well known that, in case of irrational relation

$$\gamma = \frac{\lambda_1}{\lambda_2}$$

all the trajectories prove to be solid everywhere and the measure m transitive. Besides, by following Markov (2) it is easy to demonstrate that with the irrationality y , the dynamic system is strictly ergodic, i.e. it contains one sole ergodic set E whose points have a measure as their own

$$\mu_E = cm,$$

where c is the constant. It is natural to maintain that the motion over the two-dimensional torus under conditions (1) "Generally speaking" possess all the enumerated new aspects, and this assertion is the application of the mentioned principle of neglecting the incidents when some final system of functionals (in the given case λ_1 and λ_2) obtains values from some set of measure zero (in the present case out of the set of points (λ_2, λ_1) with rational relation γ).

I managed to go somewhat further in the note (3). Namely, I demonstrated that in the assumption that such $c > 0$ and $h > 0$ exist, the inequality takes place for all integers r and s:

$$|r - s\gamma| \geq ch^s, \qquad (2)$$

and equations of motion can be reduced by analytical transformation of coordinates to the form

$$\frac{dx_1}{dt} = \lambda_1, \frac{dx_2}{dt} = \lambda_2. \qquad (3)$$

C

As is well known in the Diophantine approximation theory, the condition (2) has been carried out (with proper c and h) for almost all irrationalities y. In that way, with the exception of cases when y, by means of fractions r/s approaches "abnormally well" the analytical dynamic system with the integral invariant on the torus T^2 under conditions (1), inevitably turns out to be assuming only nearly periodic and evenmmore specially "conventionally periodic" motions with two independent frequencies λ_1 and λ_2.

Many problems of classical mechanics with two degrees of freedom are known (s = 2, n = 4), in which, because of the availability of two well-defined first integrals I_1 and I_2 on all Ω^4, the four-dimensional variety Ω^4 disintegrates, with the exception of some exceptional varieties of not more than three dimensions, into two-dimensional varieties

$$L^2_{C_1C_2} = L^2(I_1 = C_1,\ I_2 = C_2).$$

Since four equations are executed in stationary points

$$\frac{\partial H}{\partial q_1} = \frac{\partial H}{\partial q_2} = \frac{\partial H}{\partial p_1} = \frac{\partial H}{\partial p_2} = 0,$$

so in the event of analytical function H, their set on Ω^4 is not more than countable. Therefore, they can get into the variety L^2 as an exception. Hence the conclusion that almost all compact varieties L^2 are precisely tori (as orientable, compact, two-dimensional varieties, permitting a vector field without zero vectors).

As is known, problems of classical mechanicsoof the type under consideration are always integrated. Qualitative investigation of special problems of such kind (motion under the action of the gravity force along the rotation surface of triaxial ellipsoid and so on, the motion of the point along the plane under the influence of the Newtonian attraction of two fixed centers and so on) brings forward a great number of space disintegration Ω^4 primarily into tori T^2 with windings of trajectories of conditionally periodic motions with two independent frequencies λ_1 and λ_2, everywhere filling them closely. Generally speaking, everywhere there are located dense sets of tori that are broken up into closed trajectories because of the commensurability of frequencies, and, in a distinct fashion, not more than three-dimensional special varieties, on which, in particular, are placed stationary points and so-called "asymptotic" motions. Investigation of such integrated problems offersmmany interesting examples of sufficiently complex splittings of phrase space Ω into ergodic sets and the remainder of "irregular points" which are located on the trajectories of asymptotic motions*).

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*) In connection with this I shall remark here that a very instructive qualitative analysis of a problem about the attraction by two fixed centers, carried out in the well-known Sharl'e interpretation, proved to be incomplete and partly inaccurate and has been corrected twice (4), (5).

In the note referred to by me (3) it has been pointed out that with the exclusive irrational y (not meeting the demands of clause (2)), there is actually a number of new possibilities, sometimes quite unexpected for analaytical systems (more will be said about this later). However, these exclusive incidents do not occur in the problems referred to of classical mechanics due to a very simple cause: the transition to circular coordinates ξ_1, ξ_2 on tori T^2 and to parameters of these tori C_1, C_2 in these problems is accomplished by transformations of tangency. Therefore, the equations preserve the canonical form

$$\frac{d\xi_\alpha}{dt} = \frac{\partial}{\partial C_\alpha} H, \frac{dC_\alpha}{dt} = -\frac{\partial}{\partial \xi_\alpha} H$$

and, as the invariance of tori T^2 is obtained only in the event

$$\frac{dC_1}{dt} = \frac{dC_2}{dt} = 0,$$

so H is subject only to C_1 and C_2, which fact leads on each torus T^2, with no exceptions, to equations (3) with constants λ_1 and λ_2.

Therefore, a concrete significance for classical mechanics of the analysis cited by me and pertaining to dynamic systems on T^2 is subject to whether there exist sufficiently important examples of canonical systems with two degrees of freedom not integrated by classical methods, in which invariant (as regards transformations S^t) two-dimensional tori play an important part.

In order to be convinced that such examples do exist, we shall examine, by siding with the investigation of the vicinity of elliptic periodic motion conducted by Birkhoff (6), the system with circular coordinates q_1, q_2 and impulses p_1, p_2 for which

$$H(q, p) = W(p).$$

The equations of motions have the form

$$\frac{dq_\alpha}{dt} = \frac{\partial W}{\partial p_\alpha}, \quad \frac{dp_\alpha}{dt} = 0,$$

It is obvious that tori T^2_c, segregated by the conditions

$$p_1 = c_1, \quad p_2 = c_2,$$

are invariant and a provisionally periodic motion takes place on each of them

$$\frac{dq_\alpha}{dt} = \lambda_\alpha(c) = \frac{\partial}{\partial c_\alpha} W(c_1, c_2)$$

with two frequencies subordinate to C. Let us assume that the Jabobian of frequencies λ_α conforming to impulses p_α is unlike the zero:

$$\left|\frac{\partial \lambda_\alpha}{\partial p_\beta}\right| = \left|\frac{\partial^2 W}{\partial p_\alpha \partial p_\beta}\right| \neq 0. \tag{4}$$

It appears that in this case the subdivision of the investigated region of the four-dimensional space Ω^4 into two-dimensional tori T^2 is mainly stable in relation to small variations H of the form

$$H(q, p, \theta) = W(p) + \theta S(q, p, \theta).$$

In order to obtain precise formulation we shall examine region $G \subseteq \Omega^4$ determined by the condition $p \in B$, where B is the limited region on the plane of points p. By assuming the analyticity of functions W and S and under condition (4) one can prove the following : for any $\varepsilon > 0$ there exists such $\delta > 0$, that with $|\theta| < \delta$ in the dynamic system

$$\frac{dq_\alpha}{dt} = \frac{\partial}{\partial p_\alpha} H(q, p, \theta), \frac{dp_\alpha}{dt} = -\frac{\partial}{\partial p_\alpha} H(q, p, \theta)$$

the entire region G, excepting the set of measure less than ε, is composed of invariant two-dimensional tori T^2, in each of which in proper (analytically subordinated to (q, p)) circular coordinates ξ_1, ξ_2, the motion is determined by equations

$$\frac{d\xi_1}{dt} = \lambda_1, \frac{d\xi_2}{dt} = \lambda_2,$$

where λ_1 and λ_2 on each T^2 are constant, i.e. appear to be conditionally periodic with two periods.

The demonstration method consists in the tracing of the destiny of initial tori T^2_c with frequencies $\lambda_\alpha(c)$, satisfying the specification (2), with transformation θ being traced, and it is being established that each of such tori does not disintegrate with the sufficiently small θ, but is displaced only in Ω, by preserving in itself the trajectories of conventionally periodic motions with constant frequencies λ_α.

Very likely, many listeners have already guessed that, in essence, the matter pertains to some processing of an idea about the possibility of avoiding the appearance of abnormally "small denominators" while estimating perturbed orbits. Such processing is being widely discussed in the literature on celestial mechanics. However, unlike the ordinary theory of disturbances, I obtain precise results, and not a deduction about the convergence of sequences of this or that approximation of finite order (with respect to θ). This is attained thanks to the fact that, instead of the estimate of perturbed motion with constant initials, I change the very initial conditions, so that it would be possible to consistently get into the motions with normal (in the sense of the condition (2)) frequencies λ_α, with the variation θ.

I shall make three more remarks regarding what has been said.

1. The theorem about the reducibility of motions on T^2 towards the form (3) can be demonstrated also under conditions of sufficiently high order of finite multiple differentiability of functions F_α and M (naturally, with ap-

propriate reduction of inference). The theorem about the preservation of tori in Ω^4, on the contrary, apparently requires, if not the analyticity W(p) and S(q,p, θ), then the existence of an infinite number of derivatives, subjected to some limitations as far as the order of their growth is concerned.

2. Exceptional set of measure $<\varepsilon$ specified in the second theorem in fact may turn out to be dense everywhere and, probably, of a positive measure with, as much as desired, small θ. This phenomenon is similar to "zones of instability", discovered by Birkhoff during the investigation of the environs of elliptic periodic trajectories (6).

3. In the capacity of one of the special cases to which all that has been said can be applied, one can indicate the inertial motion along the analytical surface, close to the triaxal ellipsoid.

Par. 3. <u>Do Dynamic Systems on the Solid Varieties Happen to be "Generally Speaking" Transitive and Should a Continuous Spectrum Be Considered as a "General" Occurence, and the Discrete One as an "Exceptional" ?</u>

A hypothesis about the primary value of a transitive occurence and the occurence of the continuous spectrum (intermixing), has been repeatedly expressed in connection with the "ergodic" hypotheses in physics. In their application to canonical systems it is natural to attribute both of these systems only to 2s-one-dimensional invariant varieties L_h^{2s-1}, which are beingiisolated by the constancy energy requirements:

$$H = h$$

and they can be related only to the oompact L_h^{2s-1} case, as on the non-compact L_h^{2s-1} in the simplest of problems there are (usually somwhat predominant) "departing" trajectories which will be mentioned further (in Par. 4). In the event of the first hypothesis being rejected, the second one, naturally, is to be replaced not to the entire variety Ω^n (or L_h^{2s-1} in case of canonical systems), but to these ergodic sets into which Ω^n falls (by permitting, of course, to disregard ergodic sets whose sum has zero measure).

As far as applying to analytical canonical systems is concerned, one should answer both questions negatively, as the theorem about the stability of tori partitioning, stated by us in relation to systems with two degrees of freedom, is also preserved with any number of freedom degrees. If in the 2s-dimensional toroidal layer G of the phrase space Ω^{2s}

$$H(q, p, \theta) = W(p) + \theta S(q, p, \theta),$$

so with $\theta = 0$ this layer in an apparent manner splits into invariant s-dimensional tori T_p^s, on each of which the motion is conditionally periodic with s periods, whereupon in case

$$\left|\frac{\partial^2 W}{\partial p_\alpha \partial p_\beta}\right| \neq 0$$

in almost all tori T_p^s the periods are independent in the sense

$$(n, \lambda) = \sum_\alpha n_\alpha \lambda_\alpha \neq 0$$

with any integrals n_α and, therefore, the trajectories are tightly wound around the torus everywhere, the s-dimensional Lebesque measure on T^s is transitive, and the entire torus represents one ergodic set. Theorems 1 and 2 in my memorandum (22) maintain that in the described situation with small θ, this picture changes in only this respect, that some tori, corresponding to frequency systems, for which the expressions (n, λ) decrease with the ~~rise~~

$$|n| = \sqrt{\sum n_\alpha^2},$$

too rapidly, may disappear, and the majority of tori T_p^s, by preserving the character of motions occuring in them, are being somewhat displaced in Ω^{2s}, and proceeding to fill G, correct up to the act of small measure remains splitting into ergodic sets with discrete spectrum (of the special indicated nature).

In connection with this, it is interesting to note that some physicists (see, for example (7)) expressed their opinion about the hypothesis that the disintegration of Ω^{2s} into s-dimensional tori T^s that are carrying conditionally periodic motions with s periods appears to be a "general case" of canonical dynamic system without departing trajectories. This idea is apparently based only on the primary regard of a linear system and of the limited set of integrable classic problems, but, in any case, it should be noted that the demonstration methods of the above quoted theorem are substantially attatched exactly to the stratification Ω^{2s} on tori T^s and are not applicable to the stratification into tori of some other dimensionality $r > s$ or $r < s$.

It is doubtful whether the indicated hypothesis can be supported in a general form, as it is very likely that with any s there exist examples of canonical systems with s degrees of freedom and stable transitivity and intermixing on varieties L_h^{2s-1}. I have in mind the movement of a constant negative curvature over the geodesics on compact varieties V^s, i.e. dynamic systems with

$$H(q, p) = \sum_{\alpha\beta} g_{\alpha\beta}(q) p_\alpha p_\beta,$$

where the g_α coordinates on the compact varieties V^s of the constant negative curvature, and $g_{\alpha\beta}$ components of the metric tensor on V^s.

The stability of negative curvature in relation to small modifications of functions $g_{\alpha\beta}(q)$ does not require explanations. The difficulties lie only in the fact that the modification of functions $g_{\alpha\beta}(q)$ does not appear to be the only possible aspect of the modifications of functions H(q, p) and the transitivity and intermixing with $s > 2$ remain proven only for the case of constant curvature, while with the variation $g_{\alpha\beta}$ | the curvature ceases to be constant. The second difficulty in cases s=2 for which the transitivity has been proven, and with variable curvature, is eliminated. The first difficulty does not exist if one restricts oneself to functions H(q, p) of the form

$$H(q, p) = U(q) + \sum_{\alpha\beta} g_{\alpha\beta}(q) p_\alpha p_\beta \qquad (2)$$

(with which classical mechanics is dealing), as with the transition to the new metric systems of the form (2) are reduced to the systems of form (1),

If one recalls what has been previously said about the inertial motion on the surfaces close to the triaxial ellipsoid, we come to the conclusion that already in the simplest of problems of classical mechanics one has to consider stable cases and therefore in entitled to equal and basic attention, to the lesser extent in the two considered cases, one of which is connected with the transitivity on the varieties of constant energy and continuous spectrum, and another one with the absence of transitivity and principally with a discrete spectrum.

I do not know similar results pertaining to the stability of this or other general type of behavior of non-canonical dynamic systems with integral invariant and compact Ω^n.

Par. 4. Some Remarks About the Non-Compact Case

The possibility that there exist trajectories departing from any compact part Ω with $t \to \infty$, or with $t \to \infty$, appears to be a peculiarity of non-compact case. I shall expound here some general aspects of ergodic theory, suitable for any continuous S^t flows in the locally-compact spaces Ω. As the unilateral departure into infinity is only possible for trajectories forming set measures zero, the departing point x is determined at once by the requirement that such T for any compact K should exist so that all the points S_x^t c $|t| > T$ are situated outside of K. We shall designate the departing points set by $\partial\Omega'$. For the purposes of detailed analysis of concrete classical dynamic systems it is advisable to construct an "individual ergodic theory", not in the purely metric variant set forth in Hopf's book (9), but conforming to earlier work by Hopf and Stepanov (10), (11), and in some items directly following Krylov's and Bogolyubov's memoir's presentation, though the latter implies a compact case.

With such presentation, the idea of a regular point remains basic, just as in the compact case. Point x has such a name if the invariant measure μ, possessing the following properties exists for it:

1. $\mu(\Omega - \bar{I}_x) = 0$, where $\bar{I}_x$ is the closing of a trajectory passing through x.

2. $\mu(V_y) > 0$ for any vicinity V_y of the point $y \in I_x$.

3. For any two continuous functions f(x) and g(x) different from zero only on the compact sets

$$\lim \frac{\int_a^T f(S_x^t)\,dt}{\int_a^T g(S_x^t)\,dt} = \frac{\int_\Omega f\,d\mu}{\int_\Omega g\,d\mu},$$

if only

$$\int_\Omega g\,d\mu \neq 0.$$

4. The measure μ is transitive.

As the normalization requirement is absent, the measure μ is determined by a point correct only within the constant factor. Nevertheless, we shall mark it μ_x and shall name it an "individual measure" of the point x. Because of this, a small change is being introduced in the definition of ergodic sets: two points x and x' are related to one ergodic set if their individual measures coincide in the sense of coincidence correct to within a constant multiplier. Thus all sets Ω' of regular points are represented in a form of ergodic sets $\Omega' = \Sigma \varepsilon$.

Measures $\mu_{\mathcal{E}}$, naturally, are also determined now by ergodic set only correct to within the constant multiplier.

An individual ergodic theorem maintains that

$$\Omega = \Omega' + \Omega'' + N, \quad \text{where} \quad \lambda(N) = 0$$

for any invariant measure λ. For us it is essential, however, primarily, that always

$$m(N) = 0$$

Any transitive invariant measure μ is either measure $\mu_{\mathcal{E}}$ of some ergodic set $\mathcal{E}$, or it has the form

$$\mu(A) = r_I(A \cap I),$$

where r_I is a "temporary" measure on departing trajectory I. In contrast to the second trivial case, it is natural to call measures of the first type ergodic, as they are in line with the set ε_μ with

$$\mu_{\varepsilon_\mu} = \mu.$$

These considerations which, in the event of a compact Ω can be cited in favor of the opinion that the compact dynamic system of a "general form" is transitive in the application to non-compact dynamic systems, bring forward a hypothesis that, "generally speaking", one out of two cases takes place: the system is either dissipative (i.e. almost all its points are departing), or the measure m is ergodic (apparently, in the second case the departing points establish only a set of measure zero).

Sometimes this hypothesis is also applied to separate classical problems in such a form: if a certain number of first integrals exist in a given problem and there is no reason to expect the discovery of new ones, it is considered probable that a transitivity will take place on the varieties determined by the designation of valued of well-known first integrals. In corroboration with such practice one may cite such a remark that, according to Hedlund and Hopf, this alternative always occurs for motions conforming to the geodesics of the constant negative curvature.

If it has been known beforehand that there exists a set of positive measure of departing points, a hypothesis arises that the system is dissipative in conformity with what has been said. Seemingly, Birkhoff's assumption about the dissipative character of the three-body problem has been based on such a kind of deliberation.

However, it seems probable that with methods specified in Par. 3 for canonical systems, one can construct examples of stable, simultaneous location in Ω^{2s} of the dissiaptive portion of positive measure and positive domain G, filled in the main with s-dimensioned invariant tori.

I shall note that out of more elementary matters, specialists in the qualitative theory of differential equations are little occupied by concrete problems about departing trajectories of various special types. As a striking example of this is the circumstance that the refutation of Chazy's assertions about the impossibility of "interchange" and "capture" in the three-body problem (17), (18), was at first reached in a hard (and without accurate appraisals of errors logically unconvincing!) way of numerical integration (Becker (19), Schmidt (20)), and only recently the example of "capture" was constructed by Sitnikov very simply and almost without calculations (21).

Par. 5. Transitive Measures, Spectra and Eigen-Functions of Analytical Systems.

Let us call measure μ in Ω^n an analytical one, if it can be defined in the form

$$\mu(A) = \int_{V^k \cap A} f(\xi) d\xi_1 \ldots d\xi_k,$$

where V^k is some locally closed analytical variety of some dimensions $k \leq n$ in Ω^n, and the function f from coordinates ξ_α on V^k (analytically dependent on coordinates x_α in Ω^n) is analytical.

The variety V^k is identically determined by measure μ (if it is not an identical zero). Therefore, the number k can be also designated by the dimensionality of measure μ.

Transitive measures will be of special interest to us.

In this case the variety V^k should be invariant. Invariant varieties of one and the same dimension do not intersect each other, and those of different dimension can only be wholly inserted each other (of lesser dimension into the larger one). Each invariant variety carries in itself not more than one transitive measure. In virtue of what has been said, the system of analytical transitive measures has a comparatively visible structure. Only analytical transitive measures were well-known up to the recent time. Only recently, Grabar' (13), by constructing an analytical analogue of Markov's example (analytical non-reducible, but not strictly ergodic dynamic system) showed an example of non-analytical transitive measure in the analytical system. However, it is possible that the sum of all non-analytic ergodic sets is always neglected in the sense of the basic measure m.

Ergodic sets are determined identically by their measures μ_ε, which are transitive by their very definition.

As to ergodic sets corresponding to analytical transitive measures (not leading to measure μ_ε of one trajectory) we note only that, in case of the analyticity of measure μ_ε, the ergodic set is situated on the medium V^r of measure μ_ε and it is there everywhere dense, though in some simple classical examples the remainder $V^{r-\varepsilon}$ can be also dense everywhere on V^r.

Spectral characteristics of transitive measures on analytical systems have not been studied enough.

Discrete spectra so far have been obtained only with a finite base of independent frequencies

$$\lambda_1, \lambda_2, \ldots, \lambda_z,$$

moreover, the number of independent frequencies for analytical measures coincides with dimensionality in all known cases.

A continuous spectrum has been fully defined only recently by Gel'fand and Fomin (14), (15) for some cases of notions conforming to the geodesics on the surfaces of constant negative curvature. In these cases it proved to be countably-multiple Lebesque.

A possibility has not been ruled out that only these cases (discrete spectrum with a final number of independent frequencies and the countably-multiple Lebesque) provw to be possible for analytical transitive measures or that only they are in this or another sense general, typical cases.

For non-analytic transitive measures their completely arbitrary structure is represented as more probable. This could have been definite, if an analytical analogue of the Kakutani theorem (16) would not have been established pertaining to the isometric flow into that of the continuous dynamic system.

In connection with the eigen-functions, we shall dwell only on the example of analytical dynamic system on the two-dimensional torus T^2 with discrete spectrum and with everywhere totally disconnected eigen-functions. It is true that this example is connected with abnormally well approximated rational fractions r/s ratio $\gamma = \lambda_1/\lambda_2$ of mean frequencies, and beacause of its very origin rather indicates this fact that we are dealing not with a typical, but exceptional phenomenon.

In order to clear up this matter more comprehensively, we shall investigate again the equations of motion over the two-dimensional torus, by introducing into them the parameter Θ, variable in some kind of limit $[\theta_1; \theta_2]$:

$$\frac{dx_\alpha}{dt} = F_\alpha(x_1, x_2, \theta).$$

We shall assume that the functions $F_\alpha(x_1, x_2, \theta)$ are analytic. It is obvious that the ratio of mean frequencies $\gamma(\theta)$ will analytically depend on θ. If $\gamma(\theta)$ is not constant, the set R of those θ for which the system can be analytically transformed into a form

$$\frac{d\xi_\alpha}{dt} = \lambda_\alpha,$$

will take up almost the entire segment $[\theta_1, \theta_2]$. Eigen functions

$$\varphi_{mn} = e^{i(m\lambda_1 + n\lambda_2)}$$

upon being returned to the initial coordinates x_1, x_2 will serve for $0 \in R$ as analytical functions from x_1 and x_2. However, generally speaking, even on R they will be in this set with respect to θ totally disconnected, and also this discontinuity cannot be destroyed by the rejection from R of the set of measure zero. These circumstances are more significantly essential than this in that $\varphi_{mn}(x_1, x_2, 0)$ can be determined also in some points of the residual set $[0_1, 0_2]-R$ of measure zero at the expense of the assumption of their non-analyticity and discontinuity with respect to x_1 and x_2.

It is possible that the relation of $\varphi_{mn}(x_1, x_2, 0)$ the parameter θ on R is attributed to the class of functions of the type pf monogenic Bord functions (24) and, in spite of totally disconnected character, it admits the investigation with proper analytical means.

Conclusion

I shall consider my object achieved if I succeed in convincing the audience that, despite great difficulties encountered and a limited nature of results obtained, the problem of utilizing general ideas of modern ergodic theory for the analysis of qualitative character of motion in analytic and specifically canonical dynamic systems deserves great attention on the part of resesrchers capable of embracing those manifold connections which are revealed here with the most different branches of mathematics. In conclusion, I wish to thank the organizational committee of the Congress for the opportunity given to me to read this report and for the kind assistance in mimeographing the summary with formulae and literary references, and to all assembled here for the attention shown to me during this last day of our work when everyone was weary of listening to an enormous amount of material of previous days.

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